MANY-BODY WIGNER QUANTUM SYSTEMS

T. D. Palev* and N. I. Stoilova*

International Centre for Theoretical Physics, 34100 Trieste, Italy

Abstract. We present examples of many-body Wigner quantum systems. The position and the momentum operators $\mathbf{R}_A$ and $\mathbf{P}_A$, $A = 1, \ldots, n+1$, of the particles are noncanonical and are chosen so that the Heisenberg and the Hamiltonian equations are identical. The spectrum of the energy with respect to the centre of mass is equidistant and has finite number of energy levels. The composite system is spread in a small volume around the centre of mass and within it the geometry is noncommutative. The underlying statistics is an exclusion statistics.

1. Introduction

In the present paper we continue the study of the Wigner quantum systems (WQSs), initiated in [1-4]. Our interest in the subject is stimulated from the observation that some WQSs show atractive features, which cannot be achieved in the frame of the canonical quantum mechanics. The Wigner quantum system (WQS) from [1] (two noncanonical, nonrelativistic point particles interacting via harmonic potential), for instance, exhibits a quark like structure: the composite system has finite size, both constituents are bound to each other; moreover the geometry is noncommutative, the different coordinates do not commute. Another example following from [3,4]: two spinless particles, curling around each other, produce an orbital (internal angular) momentum 1/2.

Here we extand the results of [1] to the case of any number of particles. For definiteness we consider $n+1$ particles of the same mass m with a Hamiltonian

$$H_{tot} = \sum_{A=1}^{n+1} \frac{(\mathbf{P}_A)^2}{2m} + \frac{m\omega^2}{2(n+1)} \sum_{A<B=1}^{n+1} (\mathbf{R}_A - \mathbf{R}_B)^2, \tag{1.1}$$

where $\mathbf{R}_A = (R_{A1}, R_{A2}, R_{A3})$ and $\mathbf{P}_A = (P_{A1}, P_{A2}, P_{A3})$, $A = 1, \ldots, n+1$, are the positions and the momentum operators of the particles, respectively.

* Permanent address: Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria; E-mail: palev@bgearn.acad.bg, stoilova@inrne.acad.bg

The new features of the WQSs stem from the circumstance that their position and momentum operators (*RP-operators*) do not in general satisfy the canonical commutation relations (CCRs) (bellow and throughout $[x,y]=xy-yx,\ \{x,y\}=xy+yx$):

$$[R_{Aj},P_{Bk}]=i\hbar\delta_{jk}\delta_{AB},\quad [R_{Aj},R_{Bk}]=[P_{Aj},P_{Bk}]=0,\quad j,k=1,2,3,\ \ A,B=1,\ldots,n+1. \tag{1.2}$$

In certain cases the defining relations are weaker than the CCRs (the one-dimensional oscillator of Wigner [5], the $osp(1/6)$ oscillators [3]) and therefore the canonical picture appears as a particular representation of the *RP-operators*. In other cases ([1], the $osp(3/2)$ oscillator [3,4]) the *RP-operators* do not reproduce the canonical picture. The present paper is another example of this kind.

The idea for studing such more general quantum systems belongs to Wigner [5] (see the discussions in [1-4]), who has generalized a result of Ehrenfest [6], sometimes refered to as an Ehrenfest theorem [7]. The latter states (up to ordering details, which are important, but will not appear in our considerations) that in the Heisenberg picture of the quantum mechanics the Hamiltonian (resp. the Heisenberg) equations are a unique consequence from the CCRs and the Heisenberg (resp. Hamiltonian) equations. In [5] Wigner has proved a stronger statement. He has shown that for certain interactions the Hamiltonian equations can be identical to the Heisenberg equations for position and momentum operators, which do not necessarily satisfy the canonical commutation relations. Wigner has demonstrated this on an example of a one-dimensional harmonic oscillator, studied subsequantly by several authors [8]. This observation is in the origin of our definition of a Wigner quantum system. The main point is that the position and the momentum operators $\mathbf{R}_A=(R_{A1},R_{A2},R_{A3})$ and $\mathbf{P}_A=(P_{A1},P_{A2},P_{A3}),\ A=1,\ldots,n+1$, are considered as unknown operators, which have to be defined in such a way that the Heisenberg equations

$$\dot{\mathbf{P}}_A=-\frac{i}{\hbar}[\mathbf{P}_A,H_{tot}],\quad \dot{\mathbf{R}}_A=-\frac{i}{\hbar}[\mathbf{R}_A,H_{tot}] \tag{1.3}$$

are identical with the Hamiltonian equations

$$\dot{\mathbf{P}}_A=-\frac{m\omega^2}{n+1}\sum_{B=1}^{n+1}\big(\mathbf{R}_A-\mathbf{R}_B\big),\quad \dot{\mathbf{R}}_A=\frac{\mathbf{P}_A}{m}. \tag{1.4}$$

In addition the *RP-operators* have to satisfy other natural physical requirements. On the first place, they have to be defined as hermitian operators in a Hilbert space W, the state space of the system. Next, the description should be covariant with respect to the transformations from the Galilean group G. In particular we have to define the generators of G as polynomials of $\mathbf{R}_A=(R_{A1},R_{A2},R_{A3})$ and $\mathbf{P}_A=(P_{A1},P_{A2},P_{A3}),\ A=1,\ldots,n+1$, (and to identify the generators of the space rotation group $SO(3)$ in G) so that $\mathbf{R}_A=(R_{A1},R_{A2},R_{A3})$ and $\mathbf{P}_A=(P_{A1},P_{A2},P_{A3})$ tranform as vectors. These restrictions on the *RP-operators* are in addition to those imposed from the requirement the Heisenberg equations (1.3) to be identical with the Hamiltonian equations (1.4).

Our considerations are all of the time in the Heisenberg picture. We underline that the results depend on the dynamics, since the solution for $\mathbf{R}_A,\ \mathbf{P}_A$ we are searching for hold only for the Hamiltonian (1.1).

The paper is organized as follows. In the beginning of Sect. 2 we state the postulates of a Wigner quantum system. Then accepting some natural assumptions, which hold in the canonical quantum mechanics, we separate the centre of mass variables, which are postulated to be canonical. The rest of the problem is reduced to a noncanonical $3n$-dimensional Wigner oscillator for the internal variables, i.e., the Hamiltonian reads

$$H = \sum_{\alpha=1}^{n} \Big(\frac{\mathbf{p}_\alpha^2}{2m} + \frac{m\omega^2}{2} \mathbf{r}_\alpha^2 \Big). \tag{1.5}$$

In Sect. 3 we study one possible solution for $\mathbf{r}_\alpha,\ \mathbf{p}_\alpha, \quad \alpha = 1, \ldots, n$. It is defined in terms of operators, called creation and annihilation operators (CAOs), which satisfy certain relations (see (3.8)). In 3.1 we construct a class of Fock representations of the CAOs. Each such representation space is a state space of the oscillator. It is irreducible and finite-dimensional. The set of all Fock spaces are labeled with one positive integer $p = 1, 2, \ldots$. As a result (Subsect. 3.2) the spectrum of the Hamiltonian (1.5) is equidistant and has $min(3n+1, p+1)$ different values. The related $(n+1)-$particle system has finite space dimensions; the maximal distance between any two of its constituents is $D = \sqrt{\frac{6\hbar p}{(3n-1)m\omega}}$. The internal angular momentum M of the sistem takes all integer values from 0 to n. A particular feature of the coordinate operators, corresponding to each particle, is that they do not commute with each other. Therefore, although the distances between the particles are integrals of motion, the position of each individual particle cannot be localized. The kind of noncommutative geometry, obtained in this way, holds only in a very small volume around the centre of mass. In a first approximation, namely up to additive terms proportional to $\sqrt{\hbar}$, the coordinates of all species coincide with the centre of mass coordinates (see (3.31)). In 3.3 we discuss shortly the underlying algebraical structure of the CAOs. It turns out the creation and the annihilation operators are odd generators of the orthosymplectic Lie superalgebra $sl(1/3n)$. Therefore the Fock representations of the CAOs are in fact representations of this Lie superalgebra. Section 4 is not directly related to the Wigner quantum systems. Here we describe shortly the statistics of the creation and the annihilation operator, called A-superstatistics. In particular we formulate the Pauli principle of the A-superstatistics, which identify it as one of the exclusion statistics of Haldane [9]. It turns out that the A-superstatistics is very similar to the statistics of the $g-$ons as introduced by Karabali and Nair [10]. Some possible applications of the A-superstatistics are also mentioned. We complete the paper (Sect. 5) trying to justify why do we interpret the noncanonical operators $\mathbf{R}_\alpha$ and $\mathbf{P}_\alpha$ as position and momentum operators.

2. Wigner quantum systems

To begin with we give the following definition of a Wigner quantum system. A system with a Hamiltonian

$$H_{tot} = \sum_{k=1}^{N} \frac{\mathbf{p}_k^2}{2m_k} + V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N),$$

which depends on $6N$ variables $\mathbf{r}_k$ and $\mathbf{p}_k$, $k = 1, \ldots, N$, interpreted as (Cartesian) coordinates and momenta, respectively, is said to be a Wigner quantum system if the following conditions hold:

P1. The state space W is a Hilbert space. The observables are Hermitian (selfadjoint) operators in W. The expectation value $\langle A \rangle$ of the observable A in a state ϕ is $\langle A \rangle = (\phi, A\phi)/(\phi, \phi)$.

P2. The Hamiltonian equations and the Heisenberg equations are identical (as operator equations) in W.

P3. The description is covariant with respect to the transformations from the Galilean group.

The postulate **P1** contains the very essence of any quantum description. **P2** (Wigner postulate) is weaker than the requirement the CCRs (1.2) to hold. Hence the setting is more general and, consequently, for certain interactions [1-4] the results differ from the predictions of the canonical theory. In the general case one has to care about the ordering of the operators, a problem which does not appear for the Hamiltonian (1.1). We would not go into discussions of the postulate of Galilean invariance **P3**. Here the setting is the same as in the canonical case (see, for instanse [11]). In particular it insures that the Hamiltonian and the Heisenberg equations do not prefer any origin in space and time or any direction in space. The transition probability $|(\psi, \phi)|$, $\psi, \phi \in W$ remains unchanged under the Galilean transfomations of the states, etc.

We proceed to satisfy **P1-P3** with noncanonical position and momentum operators. To this end we pass to new variables, which formally coincide with the Jacoby coordinates and momenta [12],

$$\begin{aligned} &\mathbf{R} = \frac{\sum_{A=1}^{n+1} \mathbf{R}_A}{n+1}, \quad \mathbf{P} = \sum_{A=1}^{n+1} \mathbf{P}_A, \\ &\mathbf{r}_\alpha = \frac{\sum_{\beta=1}^{\alpha} \mathbf{R}_\beta - \alpha \mathbf{R}_{\alpha+1}}{\sqrt{\alpha(\alpha+1)}}, \quad \mathbf{p}_\alpha = \frac{\sum_{\beta=1}^{\alpha} \mathbf{P}_\beta - \alpha \mathbf{P}_{\alpha+1}}{\sqrt{\alpha(\alpha+1)}}, \quad \alpha = 1, \ldots, n. \end{aligned} \tag{2.1}$$

Then, despite of the fact that $\mathbf{R}, \mathbf{P}, \mathbf{r}_\alpha, \mathbf{p}_\alpha$ are unknown operators, the Hamiltonian H_{tot} splits into a sum of a centre of mass (CM) Hamiltonian H_{CM} and an internal Hamiltonian H,

$$H_{tot} = H_{CM} + H, \tag{2.2}$$

where

$$H_{CM} = \frac{\mathbf{P}^2}{2m(n+1)}, \quad H = \sum_{\alpha=1}^{n} \Big(\frac{\mathbf{p}_\alpha^2}{2m} + \frac{m\omega^2}{2} \mathbf{r}_\alpha^2 \Big). \tag{2.3}$$

The Heisenberg equations (1.3) read in terms of (2.1):

$$\dot{\mathbf{P}} = -\frac{i}{\hbar}[\mathbf{P}, H_{tot}], \quad \dot{\mathbf{R}} = -\frac{i}{\hbar}[\mathbf{R}, H_{tot}], \quad \dot{\mathbf{p}}_\alpha = -\frac{i}{\hbar}[\mathbf{p}_\alpha, H_{tot}], \quad \dot{\mathbf{r}}_\alpha = -\frac{i}{\hbar}[\mathbf{r}_\alpha, H_{tot}], \quad \alpha = 1, \ldots, n. \tag{2.4}$$

The Hamiltonian equations (1.4) yield:

$$\dot{\mathbf{P}} = 0, \quad \dot{\mathbf{R}} = \frac{\mathbf{P}}{m(n+1)}, \quad \dot{\mathbf{p}}_\alpha = -m\omega^2 \mathbf{r}_\alpha, \quad \dot{\mathbf{r}}_\alpha = \frac{\mathbf{p}_\alpha}{m}, \quad \alpha = 1, \ldots, n. \tag{2.5}$$

The problem is to determine operators $\mathbf{R}, \mathbf{P}, \mathbf{r}_\alpha, \mathbf{p}_\alpha$ so that the postulates **P1-P3** hold. In these variables **P2** says that eqs. (2.4) have to be equivalent to eqs. (2.5). Certainly eqs. (2.4)-(2.5) are satisfied with

canonical operators (the CCRs bellow follow from (1.2), since the transformation (2.1) is a canonical one), namely

$$[R_i, r_{\alpha j}] = [P_i, r_{\alpha j}] = [R_i, p_{\alpha j}] = [P_i, p_{\alpha j}] = 0, \quad i, j = 1, 2, 3, \quad \alpha = 1, \ldots, n, \tag{2.6a}$$

$$[R_j, P_k] = i\hbar\delta_{jk}, \quad [R_j, R_k] = [P_j, P_k] = 0, \quad j, k = 1, 2, 3, \tag{2.6b}$$

$$[r_{\alpha j}, p_{\beta k}] = i\hbar\delta_{\alpha\beta}\delta_{jk}, \quad [r_{\alpha j}, r_{\beta k}] = [p_{\alpha j}, p_{\beta k}] = 0, \quad j, k = 1, 2, 3, \;\; \alpha, \beta = 1, \ldots, n. \tag{2.6c}$$

We wish to study other, dynamically dependent, solutions. Our purpose is not to determine all possible operators, satisfying **P1-P3**. Rather than that we restrict ourselves to noncanonical solutions only for the internal variables $\mathbf{r}_\alpha, \mathbf{p}_\alpha,\;\; \alpha = 1, \ldots, n$. In accordance with the canonical case, we accept

Assumption 1. The CM variables commute with the internal variables, i.e., Eqs. (2.6a) hold.

Under this assumption Eqs. (2.4)-(2.5) resolve into two independent groups, the first one depending only on the CM position and momentum operators:

$$\text{CM Hamiltonian Eqs.} \qquad \dot{\mathbf{P}} = 0, \qquad \dot{\mathbf{R}} = \frac{\mathbf{P}}{m(n+1)}, \tag{2.7a}$$

$$\text{CM Heisenberg Eqs.} \qquad \dot{\mathbf{P}} = -\frac{i}{\hbar}[\mathbf{P}, H_{CM}], \quad \dot{\mathbf{R}} = -\frac{i}{\hbar}[\mathbf{R}, H_{CM}]. \tag{2.7b}$$

The second group depends only on the internal variables ($\alpha = 1, \ldots, n$):

$$\text{Internal Hamiltonian Eqs.} \qquad \dot{\mathbf{p}}_\alpha = -m\omega^2\mathbf{r}_\alpha, \qquad \dot{\mathbf{r}}_\alpha = \frac{\mathbf{p}_\alpha}{m}, \tag{2.8a}$$

$$\text{Internal Heisenberg Eqs.} \qquad \dot{\mathbf{p}}_\alpha = -\frac{i}{\hbar}[\mathbf{p}_\alpha, H], \quad \dot{\mathbf{r}}_\alpha = -\frac{i}{\hbar}[\mathbf{r}_\alpha, H]. \tag{2.8b}$$

With the next assumption we solve equations (2.7).

Assumption 2. The center of mass coordinates and momenta are canonical, they satisfy Eqs. (2.6b).

Consequently the centre of mass behaves as a free canonical point particle with a mass $m(n+1)$. Thus we are left with the equations (2.8), which coincide with the Hamiltonian and the Heisenberg equations of a (noncanonical) $3n-$dimensional harmonic oscillator.

Turning to the Galilean covariance, we note that in the canonical situation the state space W caries a projective representation of G and of its Lie algebra g. It is an exact representation of the central extension $\hat{g}$ of g with the generator of the total mass of the system. As in the canonical quantum mechanics, we accept the following identification between the generators of g and some of the observables of the $(n+1)-$particle system:

Assumption 3.

1^0 The angular momentum operatots $\mathbf{J} = \mathbf{L} + \mathbf{M}$ are generators of the algebra $so(3)$ of the space rotations, (2.9a)

2^0 $H_{tot} = H_{CM} + H$ is a generator of the translations in time, (2.9b)

3^0 The operators of the total momentum $\mathbf{P}$ are generators of the space translations, (2.9c)

4^0 $\mathbf{K} = \mu\mathbf{R} - \mathbf{P}t$ are generators of the accelerations. (2.9d)

In (2.9) t is the time, $\mu = m(n+1)$ is the mass of the system. This already means we have chosen a representation of $\hat{g}$ with a value μ of the mass operator (which is one of the Casimir operators). $\mathbf{L} = (L_1, L_2, L_3)$ are the operators of the angular momentum of the centre of mass,

$$L_i = \frac{1}{2\hbar} \sum_{j,k=1}^{3} \varepsilon_{ijk} \{R_j, P_k\}, \tag{2.10}$$

which generate also an $so(3)$ algebra, denoted as $so(3)_{CM}$. $\mathbf{M} = (M_1, M_2, M_3)$ are operators still to be determined. In the fixed mass representation the generators of $\hat{g}$ satisfy the commutation relations ($j,k,l = 1,2,3$) [13]:

$$[J_j, J_k] = i\varepsilon_{jkl} J_l, \quad [J_j, P_k] = i\varepsilon_{jkl} P_l, \quad [J_j, K_k] = i\varepsilon_{jkl} K_l, \quad [J_j, H_{tot}] = 0, \tag{2.11a}$$

$$[P_j, P_k] = 0, \quad [P_j, K_k] = -i\hbar\delta_{jk}\mu, \quad [P_j, H_{tot}] = 0, \tag{2.11b}$$

$$[K_j, K_k] = 0, \quad [K_j, H_{tot}] = i\hbar P_j. \tag{2.11c}$$

The Galilean covariance is to a big extent covered by the above commutation relations, which have to be satisfied together with Eqs. (2.8). In particular from (2.11) one concludes that $\mathbf{J}$, $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{K}$ transform as vectors. From (2.11) however does not follow that $\mathbf{r}_\alpha$, $\mathbf{p}_\alpha$ are vectors. This is a problem still to be solved and we will solve it in few steps.

Observe first of all that the generators of the centre of mass $\mathbf{L}$, $\mathbf{P}, \mathbf{K}$ and H_{CM} satisfy (2.11) (with L_j instead of J_j). Hence these generators define a (projective) representation of an algebra, isomorphic to g, denoted here as g_{CM}. This is a representation of a point particle with a mass μ. The operators $\mathbf{L}$ and H_{CM} generate (a representation of) the subalgebra $so(3)_{CM} \oplus u(1)_{CM} \subset g_{CM}$.

In the canonical case $\mathbf{M}$ is a vector operator, commuting with H and both $\mathbf{M}$ and H are in the enveloping algebra of $\mathbf{r}_\alpha$ and $\mathbf{p}_\alpha$, $\alpha = 1, \ldots, n$. More precisely,

$$M_i = \sum_{\alpha=1}^{n} M_{\alpha i}, \quad M_{\alpha i} = \frac{1}{2\hbar} \sum_{j,k=1}^{3} \varepsilon_{ijk} \{r_{\alpha j}, p_{\alpha k}\}. \tag{2.12}$$

Therefore also here we assume that $\mathbf{M}$ can be expressed in terms of the internal variables.

Assumption 4. The components of $\mathbf{M}$ and H are generators of $so(3)_{int} \oplus u(1)_{int}$. They are in the enveloping algebra of the internal position and momentum operators $\mathbf{r}_\alpha, \mathbf{p}_\alpha$, $\alpha = 1, \ldots, n$. $\mathbf{M}$, $\mathbf{r}_\alpha$, $\mathbf{p}_\alpha$ transform as vectors with respect to $so(3)_{int}$:

$$[M_j, M_k] = i\varepsilon_{jkl} M_l, \quad [M_j, r_{\alpha k}] = i\varepsilon_{jkl} r_{\alpha l}, \quad [M_j, p_{\alpha k}] = i\varepsilon_{jkl} p_{\alpha l}, \quad \alpha = 1, \ldots, n. \tag{2.13}$$

From Assumption 4 follows that the operators

$$\mathbf{J} = \mathbf{L} + \mathbf{M}, \quad \mathbf{P}, \quad \mathbf{K} = \mu\mathbf{R} - \mathbf{P}t, \quad H_{tot} = H_{CM} + H \tag{2.14}$$

satisfy Eqs. (2.11). Moreover, the operators $\mathbf{R}_A,\ \mathbf{P}_A,\ \mathbf{r}_\alpha,\ \mathbf{p}_\alpha, \mathbf{R},\ \mathbf{P},\ \mathbf{J},\ \mathbf{K},\ \ \mathbf{L},\ \mathbf{M}$ transform as vectors. In particular,

$$[J_j, A_k] = i\varepsilon_{jkl}A_l \ \text{ for any } \ \mathbf{A} \in \{\mathbf{R}_A,\ \mathbf{P}_A,\ \mathbf{r}_\alpha,\ \mathbf{p}_\alpha, \mathbf{R},\ \mathbf{P},\ \mathbf{J},\ \mathbf{K},\ \ \mathbf{L},\ \mathbf{M}\}. \tag{2.15}$$

In other words, **P3** is a consequence of Assumption 4. From the same assumption we conclude that g_{CM} commutes with $so(3)_{int} \oplus u(1)_{int}$ (with H being a generator of $u(1)_{int}$). Therefore the (physical) Galilean algebra g is a subalgebra of a larger (Lie) algebra $g_{CM} \oplus so(3)_{int} \oplus u(1)_{int}$. Hence given representation of g is realized in a state space W, which is a tenzor product of the canonical "free-particle" state space W_{CM} with mass μ and a module (=representation space) W_{int} of the algebra $so(3)_{int} \oplus u(1)_{int}$,

$$W = W_{CM} \otimes W_{int}. \tag{2.16}$$

The CM variables $\mathbf{R},\ \mathbf{P},\ \mathbf{L}$ are hermitian operators in W_{CM}. Therefore all operators $H_{tot},\ H_{CM},\ H,\ \mathbf{R}_A,$ $\mathbf{P}_A,\ \mathbf{r}_\alpha,\ \mathbf{p}_\alpha, \mathbf{R},\ \mathbf{P},\ \mathbf{J},\ \mathbf{K},\ \ \mathbf{L},\ \mathbf{M}$ will be Hermitian operators in W, if $\mathbf{r}_\alpha,\ \mathbf{p}_\alpha$ and $\mathbf{M}$ are Hermitian operators in W_{int}. Thus condition **P1** holds if (still the unknown operators) $\mathbf{r}_\alpha,\ \mathbf{p}_\alpha$ and $\mathbf{M}$ are hermitian operators in W_{int}.

We summarize. The $(n+1)-$particle system with a Hamiltonian (1.1) is a Wigner quantum system, i.e., the postulates **P1-P3** hold, if

P1$_{int}$. The state space W_{int} is a Hilbert space. The observables (in this case $\mathbf{r}_\alpha,\ \mathbf{p}_\alpha,\ \mathbf{M}$ and H) are Hermitian operators in W_{int}.

P2$_{int}$. The internal Hamiltonian equations (2.8a) and the internal Heisenberg equations (2.8b) are identical (as operator equations) in W_{int}.

P3$_{int}$ (= *Assumption 4*). The internal Hamiltonian H and the components of $\mathbf{M}$ are generators of $so(3)_{int} \oplus u(1)_{int}$. They are polynomials of the internal position and momentum operators $\mathbf{r}_\alpha,\ \mathbf{p}_\alpha,\ \ \alpha = 1,\ldots,n$, so that

$$[M_j, H] = 0, \quad [M_j, A_k] = i\varepsilon_{jkl}A_l, \quad A_k \in \{M_i,\ r_{\alpha i},\ p_{\alpha i} | i = 1,2,3;\ \alpha = 1,\ldots,n\}. \tag{2.17}$$

The above postulates identify $\mathbf{r}_\alpha$ and $\mathbf{p}_\alpha,\ \ \alpha = 1,\ldots,n$ as position and momentum operators of a noncanonical $3n-$dimensional Wigner quantum oscillator. We proceed to study an example of such an oscillator or, more precisely, of such oscillators, since the position and the momentum operators will have several inequivalent representations.

3. sl(1/3n) Wigner quantum systems

The problem of constructing a WQS with a Hamiltonian (1.1) has been reduced to a problem of building a Wigner quantum oscillator, namely a $3n-$dimensional noncanonical oscillator with a Hamiltonian

$$H = \sum_{\alpha=1}^{n} \Big(\frac{\mathbf{p}_\alpha^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}_\alpha^2 \Big), \tag{3.1}$$

Hamiltonian equations

$$\dot{\mathbf{p}}_\alpha = -m\omega^2 \mathbf{r}_\alpha, \ \ \dot{\mathbf{r}}_\alpha = \frac{\mathbf{p}_\alpha}{m}, \quad \alpha = 1,\ldots,n, \tag{3.2}$$

and Heisenberg equations

$$\dot{\mathbf{p}}_\alpha = -\frac{i}{\hbar}[\mathbf{p}_\alpha, H], \quad \dot{\mathbf{r}}_\alpha = -\frac{i}{\hbar}[\mathbf{r}_\alpha, H], \quad \alpha = 1,\ldots,n, \tag{3.3}$$

for which the conditions $\mathbf{P1}_{int} - \mathbf{P3}_{int}$ hold.

Eqs. (3.2) and (3.3) are compatible only if

$$[H, \mathbf{p}_\alpha] = i\hbar m\omega^2 \mathbf{r}_\alpha, \quad [H, \mathbf{r}_\alpha] = -\frac{i\hbar}{m}\mathbf{p}_\alpha. \tag{3.4}$$

In this section we introduce one particular set of Wigner quantum oscillators, which we call $sl(1/3n)$ oscillators, and investigate some of the properties of the related $(n+1)-$particle system. The reason to choose this name is of an algebraic origin and will be explained in Subsect. *3.3.*

3.1 Satisfying conditions $\mathbf{P1}_{int} - \mathbf{P3}_{int}$

Introduce in place of $\mathbf{r}_\alpha, \ \mathbf{p}_\alpha$ new unknown operators

$$a_{\alpha k}^\pm = \sqrt{\frac{(3n-1)m\omega}{4\hbar}} r_{\alpha k} \pm i\sqrt{\frac{(3n-1)}{4m\omega\hbar}} p_{\alpha k}, \quad k = 1,2,3, \quad \alpha = 1,2,\ldots,n. \tag{3.5}$$

For the sake of convenience we refer to $a_{\alpha k}^+$ and to $a_{\alpha k}^-$ as to creation and annihilation operators (CAOs), respectively. These operators should be not confused with Bose operators. They are unknown operators we are searching for. In terms of these operators the internal Hamiltonian (3.1) and the compatibility condition (3.4) read

$$H = \frac{\omega\hbar}{3n-1}\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\{a_{\alpha i}^+, a_{\alpha i}^-\}, \tag{3.6}$$

$$\sum_{\beta=1}^{n}\sum_{j=1}^{3}[\{a_{\beta j}^+, a_{\beta j}^-\}, a_{\alpha i}^\pm] = \mp(3n-1)a_{\alpha i}^\pm, \quad i = 1,2,3, \quad \alpha = 1,2,\ldots,n. \tag{3.7}$$

As a solution of Eq. (3.7) we choose operators $a_{\alpha k}^\pm, \quad k = 1,2,3, \quad \alpha = 1,2,\ldots,n$, which satisfy the relations

$$[\{a_{\alpha i}^+, a_{\beta j}^-\}, a_{\gamma k}^+] = \delta_{jk}\delta_{\beta\gamma}a_{\alpha i}^+ - \delta_{ij}\delta_{\alpha\beta}a_{\gamma k}^+, \tag{3.8a}$$

$$[\{a_{\alpha i}^+, a_{\beta j}^-\}, a_{\gamma k}^-] = -\delta_{ik}\delta_{\alpha\gamma}a_{\beta j}^- + \delta_{ij}\delta_{\alpha\beta}a_{\gamma k}^-, \tag{3.8b}$$

$$\{a_{\alpha i}^+, a_{\beta j}^+\} = \{a_{\alpha i}^-, a_{\beta j}^-\} = 0. \tag{3.8c}$$

We recall that all considerations are in the Heisenberg picture. The position and the momentum operators depend on time. Hence also the CAOs depend on t. Writing the time dependence explicitly, we

obtain:

$$\text{Hamiltonian equations} \qquad \dot{a}^{\pm}_{\alpha k}(t) = \mp i\omega a^{\pm}_{\alpha k}(t), \tag{3.9}$$

$$\text{Heisenberg equations} \qquad \dot{a}^{\pm}_{\alpha k}(t) = -\frac{i\omega}{3n-1}\sum_{\beta=1}^{n}\sum_{j=1}^{3}[a^{\pm}_{\alpha k}(t), \{a^{+}_{\beta j}(t), a^{-}_{\beta j}(t)\}]. \tag{3.10}$$

The solution of (3.9) is evident,

$$a^{\pm}_{\alpha k}(t) = exp(\mp i\omega t)a^{\pm}_{\alpha k}(0) \tag{3.11}$$

and therefore if the defining relations (3.8) hold at a certain time $t = 0$, i.e., for $a^{\pm}_{\alpha k} \equiv a^{\pm}_{\alpha k}(0)$, then they hold as equal time relations for any other time t. From (3.8) it follows also that the Eqs. (3.10) are identical with Eqs. (3.9). For further references we formulate this result directly in terms of $\mathbf{r}_\alpha$ and $\mathbf{p}_\alpha$.

Conclusion 1. Within *any* representation space W_{int} of the CAOs (3.8) the Hamiltonian equations (3.2) are identical with the Heisenberg equations (3.3), i.e., the condition $\mathbf{P2}_{int}$ holds. The explicit time dependent solutions of these equations read:

$$r_{\alpha k}(t) = \sqrt{\frac{\hbar}{(3n-1)m\omega}}(a^{+}_{\alpha k}e^{-i\omega t} + a^{-}_{\alpha k}e^{i\omega t}), \quad p_{\alpha k}(t) = -i\sqrt{\frac{m\omega\hbar}{3n-1}}(a^{+}_{\alpha k}e^{-i\omega t} - a^{-}_{\alpha k}e^{i\omega t}). \tag{3.12}$$

Turning to condition $\mathbf{P3}_{int}$, we set

$$M_{\alpha j} = -i\sum_{k,l=1}^{3}\varepsilon_{jkl}\{a^{+}_{\alpha k}, a^{-}_{\alpha l}\} = -\frac{3n-1}{2\hbar}\sum_{k,l=1}^{3}\varepsilon_{jkl}\{r_{\alpha k}, p_{\alpha l}\}, \quad j = 1,2,3, \quad \alpha = 1,2,\ldots,n. \tag{3.13}$$

Then

$$[H, M_{\alpha j}] = 0, \quad [M_{\alpha j}, M_{\alpha k}] = i\sum_{l=1}^{3}\varepsilon_{jkl}M_{\alpha l}, \quad j,k,l = 1,2,3, \tag{3.14}$$

i.e., for each $\alpha = 1,2,\ldots,n$ the operators $\mathbf{M}_\alpha = (M_{\alpha 1}, M_{\alpha 2}, M_{\alpha 3})$ are generators of an $so(3)$ algebra, which we denote $so(3)_\alpha$. Eqs. (3.8) yield that any two different algebras commute,

$$[so(3)_\alpha, so(3)_\beta] = 0 \quad \forall \alpha \neq \beta = 1,\ldots,n. \tag{3.15}$$

It is straightforward to check that the operators H and $M_i = \sum_{\alpha=1}^{n} M_{\alpha i}$ satisfy Eqs. (2.17). Thus, we have

Conclusion 2. Within *any* representation space W_{int} of the CAOs (3.8) the operators H and $\mathbf{M} = (M_1, M_2, M_3)$ satisfy the condition $\mathbf{P3}_{int}$.

It remains to define the (internal) position and the momentum operators $\mathbf{r}_\alpha$ and $\mathbf{p}_\alpha$, corresponding to the CAOs (3.8), as linear Hermitian operators in a Hilbert space, which will be the internal state space W_{int}. In terms of the creation and the annihilation operators this means that the Hermitian conjugate to $a^{+}_{\alpha k}$ should be equal to $a^{-}_{\alpha k}$, i.e.,

$$(a^{+}_{\alpha k})^{\dagger} = a^{-}_{\alpha k}. \tag{3.16}$$

The CAOs (3.8) have several representations. Here, as in [1], we consider only representations which are obtained by the usual Fock space technique. The irreducible Fock representations are labelled by one

non-negative integer $p = 1, 2, \ldots$, called an order of the statistics. To construct them assume that the corresponding representation space $W(n,p)_{int}$ contains (up to a multiple) a cyclic vector $|0\rangle$, such that

$$a^-_{\alpha i}|0\rangle = 0, \quad a^-_{\alpha i}a^+_{\beta j}|0\rangle = p\delta_{\alpha\beta}\delta_{ij}|0\rangle, \quad i,j = 1,2,3, \quad \alpha = 1,2,\ldots,n. \tag{3.17}$$

The above relations are enough for reconstructing the representation space $W(n,p)_{int}$. Let

$$\Theta \equiv \{\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \ldots, \theta_{n1}, \theta_{n2}, \theta_{n3}\}.$$

Since $(a^+_{\alpha i})^2 = 0$, from (3.17) one derives that the set of all vectors

$$|p;\Theta\rangle \equiv |p;\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \ldots, \theta_{n1}, \theta_{n2}, \theta_{n3}\rangle = \sqrt{\frac{(p - \sum_{\alpha=1}^n \sum_{i=1}^3 \theta_{\alpha i})!}{p!}} \prod_{\alpha=1}^n \prod_{i=1}^3 (a^+_{\alpha i})^{\theta_{\alpha i}}|0\rangle$$

$$\equiv \sqrt{\frac{(p - \sum_{\alpha=1}^n \sum_{i=1}^3 \theta_{\alpha i})!}{p!}} (a^+_{11})^{\theta_{11}}(a^+_{12})^{\theta_{12}}(a^+_{13})^{\theta_{13}}(a^+_{21})^{\theta_{21}}(a^+_{22})^{\theta_{22}} \ldots (a^+_{n1})^{\theta_{n1}}(a^+_{n2})^{\theta_{n2}}(a^+_{n3})^{\theta_{n3}}|0\rangle \tag{3.18}$$

with

$$\theta_{\alpha i} = 0,1 \quad \text{and} \quad k \equiv \sum_{\alpha=1}^n \sum_{i=1}^3 \theta_{\alpha i} \leq p, \tag{3.19}$$

constitute an orthonormal basis in $W(n,p)_{int}$ with respect to the scalar product, defined in the usual way with "bra" and "ket" vectors and $\langle 0|0\rangle = 1$. We underline that the product of the multiples in $\prod_{\alpha=1}^n \prod_{i=1}^3 (a^+_{\alpha i})^{\theta_{\alpha i}}$ is ordered as indicated in (3.18)

Let $|p;\Theta\rangle_{\pm\alpha i}$ be a vector, obtained from $|p;\Theta\rangle$ after a replacement of $\theta_{\alpha i}$ with $\theta_{\alpha i} \pm 1$. Then the transformation of the basis under the action of the CAOs read:

$$a^-_{\alpha i}|p;\Theta\rangle = \theta_{\alpha i}(-1)^{\theta_{11}+\ldots+\theta_{\alpha,i-1}} \sqrt{p - \sum_{\beta=1}^n \sum_{j=1}^3 \theta_{\beta j} + 1}\, |p;\Theta\rangle_{-\alpha i}, \tag{3.20a}$$

$$a^+_{\alpha i}|p;\Theta\rangle = (1 - \theta_{\alpha i})(-1)^{\theta_{11}+\ldots+\theta_{\alpha,i-1}} \sqrt{p - \sum_{\beta=1}^n \sum_{j=1}^3 \theta_{\beta j}}\, |p;\Theta\rangle_{\alpha i}. \tag{3.20b}$$

The next conclusion is easily verified.

Conclusion 3. The operators $\mathbf{r}_\alpha$, $\mathbf{p}_\alpha$, $\mathbf{M}$ and H are Hermitian operators within every Hilbert space $W(n,p)_{int}$, $p = 1, 2, \ldots$.

Remark. The requirement $\sum_{\alpha=1}^n \sum_{i=1}^3 \theta_{\alpha i} \leq p$ can be skipped. In such a case one is getting a larger representation space, which carries an indecomposible representation of the CAOs. The hermiticity condition (3.16), however, cannot be satisfied in this larger space. If p is not a positive integer, (3.16) also cannot be fulfiled in a space with a positive definite metric.

We have satisfied all requirements of conditions $\mathbf{P1}_{int} - \mathbf{P3}_{int}$. Hence within each state space $W(n,p)_{int}$ the $sl(1/3n)-$oscillator is a Wigner quantum oscillator and the related $(n+1)-$particle system is a Wigner quantum system with a state space

$$W(n,p) = W_{CM} \otimes W(n,p)_{int}. \tag{3.21}$$

3.2 Properties of the $sl(1/3n)$ quantum systems

3.2.1 Spectrum of the internal Hamiltonian

Note, first of all, that the internal state space $W(n,p)_{int}$ is finite-dimensional. From (3.6) and (3.20) one concludes that the internal Hamiltonian H is diagonal in the basis (3.18),

$$H|p;\Theta\rangle = \frac{\omega\hbar}{3n-1}\big(3np-(3n-1)k\big)|p;\Theta\rangle, \tag{3.22}$$

where according to (3.19)

$$k = 0,1,2,\ldots,min(3n,p). \tag{3.23}$$

Therefore the internal energy of the system takes $min(3n,p)+1$ different values:

$$E_k = \frac{\omega\hbar}{3n-1}\Big(3np-(3n-1)k\Big), \quad k=0,1,2,\ldots,min(3n,p). \tag{3.24}$$

As in the canonical oscillator the energy spectrum is equidistant. To each energy E_k there correspond $\binom{3n}{k}$ (linearly independent) states, namely all basis vectors $|p;\Theta\rangle$ with fixed value of k.

3.2.2 Internal angular momentum

The internal state space $W(n,p)_{int}$ carries a reducible representation of each $so(3)_\alpha$. The angular momentum of each oscillating "particle" is either 0 or 1:

$$\mathbf{M}_\alpha^2|p;\Theta\rangle = 0, \text{ if } \theta_{\alpha1}=\theta_{\alpha2}=\theta_{\alpha3} \text{ and } \mathbf{M}_\alpha^2|p;\Theta\rangle = 2|p;\Theta\rangle \text{ otherwise.} \tag{3.25}$$

Each basis vector $|p;\Theta\rangle$ is an eigenvector of the square of the intermal angular momentum $\mathbf{M}^2$:

$$\mathbf{M}^2|p;\Theta\rangle = M(M+1)|p;\Theta\rangle, \quad M=1,2,\ldots,n, \tag{3.26}$$

i.e., the internal angular momentum of the composite $(n+1)-$particle system takes all integer values between 0 and n. This conclusion holds for any representation of the CAOs. The multiplicity of each individual value of $\mathbf{M}^2$ depends, however, on the order of the statistics p, namely, on the representation.

3.2.3 Geometry and space size of the system

Let us consider first the $n-$dimensional Wigner oscillator as such, independantly of the initial $(n+1)-$particle system. In order to avoid confusions, we refer to the constituents of the oscillator as to oscillating "particles" (or simply "particles").

It is straightforward to check that the square of the radius vector $\mathbf{r}_\alpha^2$ of each "particle" commutes with the (internal) Hamiltonian and, moreover, all operators $\mathbf{r}_\alpha^2$ commute with each other,

$$[H,\mathbf{r}_\alpha^2]=0, \quad [\mathbf{r}_\alpha^2,\mathbf{r}_\beta^2]=0, \quad \alpha,\beta=1,\ldots,n. \tag{3.27}$$

Hence all operators $\mathbf{r}_\alpha^2$ can be simultaneously diagonalized. The basis vectors $|p;\Theta\rangle$ are eigenvectors of these operators:

$$\mathbf{r}_\alpha^2|p;\Theta\rangle = \frac{\hbar}{(3n-1)m\omega}\Big(3p-3k+\sum_{i=1}^{3}\theta_{\alpha i}\Big)|p;\Theta\rangle, \quad \alpha=1,\ldots,n. \tag{3.28}$$

The latter indicates that the "particles" move along spheres with radiuses

$$|r_\alpha| = \sqrt{\frac{\hbar\Big(3p-3k+\sum_{i=1}^{3}\theta_{\alpha i}\Big)}{(3n-1)m\omega}}, \quad k=0,1,2,\ldots,min(3n,p). \tag{3.29}$$

Setting in (3.29) $k=0$, one obtains the maximal radius. Hence the spatial size of the oscillator (its diameter) is

$$d = 2\sqrt{\frac{3\hbar p}{(3n-1)m\omega}}. \tag{3.30}$$

The different "particles" can stay simultaneously on spheres with different radiuses. The positions of the "particles" on the spheres, however, cannot be localized, since the coordinate operators do not commute with each other, $[r_{\alpha i}, r_{\alpha j}] \neq 0, \quad i\neq j = 1,2,3$. The geometry of the oscillator is noncommutative.

Let us turn to the $(n+1)-$particle system. The expressions of $\mathbf{R}_A$ and $\mathbf{P}_A$ in terms of the Jacoby variables and also in terms of the CAOs read:

$$\begin{aligned}
\mathbf{R}_A =& \mathbf{R} - \sqrt{\frac{A-1}{A}}\,\mathbf{r}_{A-1} + \sum_{\alpha=A}^{n}\sqrt{\frac{1}{\alpha(\alpha+1)}}\,\mathbf{r}_\alpha \\
=& \mathbf{R} - \sqrt{\frac{\hbar(A-1)}{(3n-1)Am\omega}}\,(\mathbf{a}_{A-1}^+ + \mathbf{a}_{A-1}^-) + \sum_{\alpha=A}^{n}\sqrt{\frac{\hbar}{(3n-1)\alpha(\alpha+1)m\omega}}\,(\mathbf{a}_\alpha^+ + \mathbf{a}_\alpha^-),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\mathbf{P}_A =& \frac{1}{n+1}\mathbf{P} - \sqrt{\frac{A-1}{A}}\,\mathbf{p}_{A-1} + \sum_{\alpha=A}^{n}\sqrt{\frac{1}{\alpha(\alpha+1)}}\,\mathbf{p}_\alpha \\
=& \frac{1}{n+1}\mathbf{P} + i\sqrt{\frac{\hbar m\omega(A-1)}{(3n-1)A}}\,(\mathbf{a}_{A-1}^+ - \mathbf{a}_{A-1}^-) - i\sum_{\alpha=A}^{n}\sqrt{\frac{\hbar m\omega}{(3n-1)\alpha(\alpha+1)}}\,(\mathbf{a}_\alpha^+ - \mathbf{a}_\alpha^-).
\end{aligned} \tag{3.32}$$

Therefore also in this case the geometry is noncommutative. A new, somewhat unexpected feature here is that the distance operators between the particles do not commute, namely in general

$$[(\mathbf{R}_A-\mathbf{R}_B)^2, (\mathbf{R}_C-\mathbf{R}_D)^2] \neq 0, \ \text{ if } \ (A,B)\neq(C,D). \tag{3.33}$$

The only square-distance operator, which is diagonal in the basis (3.18), is $(\mathbf{R}_1-\mathbf{R}_2)^2$. From the general expression (3.31) we obtain

$$(\mathbf{R}_1-\mathbf{R}_2)^2 = 2\mathbf{r}_1^2 = \frac{2\hbar}{(3n-1)m\omega}\sum_{i=1}^{3}\{a_{1i}^+, a_{1i}^-\}. \tag{3.34}$$

Therefore

$$(\mathbf{R}_1-\mathbf{R}_2)^2|p;\Theta\rangle = \frac{2\hbar}{(3n-1)m\omega}\Big(3p-3k+\sum_{i=1}^{3}\theta_{1i}\Big)|p;\Theta\rangle. \tag{3.35}$$

Hence the spectrum of $|\mathbf{R}_1 - \mathbf{R}_2| \equiv \sqrt{(\mathbf{R}_1 - \mathbf{R}_2)^2}$ reads:

$$\sqrt{\frac{2\hbar}{(3n-1)m\omega}\Big(3p - 3k + \sum_{i=1}^{3}\theta_{1i}\Big)}, \quad k = 0, 1, 2, \ldots, min(3n, p). \tag{3.36}$$

In particular the maximal distance D between the first and the second particles is

$$D = \sqrt{\frac{6\hbar p}{(3n-1)m\omega}}. \tag{3.37}$$

Since both H and $(\mathbf{R}_1 - \mathbf{R}_2)^2$ are diagonal operators, they commute,

$$[H, (\mathbf{R}_1 - \mathbf{R}_2)^2] = 0, \tag{3.38}$$

and therefore the distance between the first and the second particles is preserved in time, it is an integral of motion.

It is natural to expect that the spectrum of $|\mathbf{R}_A - \mathbf{R}_B| \equiv \sqrt{(\mathbf{R}_A - \mathbf{R}_B)^2}$ for any $A \neq B \quad A, B = 1, \ldots, n+1$ is the same as those of $|\mathbf{R}_1 - \mathbf{R}_2|$. In particular the maximal distance between the particles with numbers A and B should be D. Whether this is, however, the case is not so easy to see. The point is that all our construction is very asymmetrical, it depends on the way one is numbering the particles. In particular the Jacoby variables (2.1) and hence also the related CAOs (3.5) do depend strongly on the fixed numbering. If one is renumbering the position and the momentum operators, setting

$$\tilde{\mathbf{R}}_\alpha = \mathbf{R}_{\sigma(\alpha)}, \;\; \tilde{\mathbf{P}}_\alpha = \mathbf{P}_{\sigma(\alpha)} \quad \text{with } \; \sigma \in S_n \;\text{ being any permutation } \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ \sigma(1), & \sigma(2), & \sigma(3), & \ldots, & \sigma(n) \end{pmatrix},$$

this will lead to new creation and annihilation operators $\tilde{a}_{\alpha i}^{\pm}$ (see (3.5)) and hence in principle to a new representation space according to (3.17).

In the following we show that the representation (and the representation space) remains the same, when renumbering the particles. We diagonalize also $(\mathbf{R}_A - \mathbf{R}_B)^2$ and show that its spectrum is the same as of $(\mathbf{R}_1 - \mathbf{R}_2)^2$ (see (3.35)). To this end we first formulate a simple proposition, which proof is straightforward.

Proposition 1. Let S be any $n \times n$ symmetric orthogonal matrix: $S^T = S, \;\; S^T S = 1$. Then

(a) The operators

$$\tilde{a}_{\alpha i}^{\pm} = \sum_{\beta=1}^{n} S_{\beta\alpha} a_{\beta i}^{\pm} \tag{3.39}$$

satisfy (3.8);

(b)

$$H = \frac{\omega\hbar}{3n-1}\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\{a_{\alpha i}^{+}, a_{\alpha i}^{-}\} = \frac{\omega\hbar}{3n-1}\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\{\tilde{a}_{\alpha i}^{+}, \tilde{a}_{\alpha i}^{-}\}; \tag{3.40}$$

(c) If Eqs. (3.17) hold, then

$$\tilde{a}_{\alpha i}^{-}|0\rangle = 0, \quad \tilde{a}_{\alpha i}^{-}\tilde{a}_{\beta j}^{+}|0\rangle = p\delta_{\alpha\beta}\delta_{ij}|0\rangle, \quad i, j = 1, 2, 3, \quad \alpha = 1, 2, \ldots, n. \tag{3.41}$$

(d) Let $P \equiv P(a_{11}^+, a_{12}^+, a_{13}^+, a_{21}^+, a_{22}^+, a_{23}^+, \ldots, a_{n1}^+, a_{n2}^+, a_{n3}^+)$ be any polynomial of the creation operators and $\tilde{P} \equiv P(\tilde{a}_{11}^+, \tilde{a}_{12}^+, \tilde{a}_{13}^+, \tilde{a}_{21}^+, \tilde{a}_{22}^+, \tilde{a}_{23}^+, \ldots, \tilde{a}_{n1}^+, \tilde{a}_{n2}^+, \tilde{a}_{n3}^+)$. If $[H, P] = 0$, then $[H, \tilde{P}] = 0$.

(e) If $\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(a_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle$ is an eigenvector of the operator P with an eigenvalue $c(p,\Theta)$, i.e.,

$$P\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(a_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle = c(p,\Theta)\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(a_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle, \tag{3.42}$$

then $\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle$ is an eigenvector of $\tilde{P}$, corresponding to the same eigenvalue $c(p,\Theta)$:

$$\tilde{P}\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle = c(p,\Theta)\prod_{\alpha=1}^{n}\prod_{i=1}^{3}(\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}}|0\rangle. \tag{3.43}$$

We proceed to find the transformatons of the CAOs under permutations of the Cartesian coordinates $\mathbf{R}_{\alpha i}$ and momenta $\mathbf{P}_{\alpha i}$. From (2.1) and (3.5) one derives

$$a_{\alpha k}^{\pm} = \sqrt{\frac{(3n-1)m\omega}{4\hbar\alpha(\alpha+1)}}\left\{\sum_{\beta=1}^{\alpha}(R_{\beta k} - \alpha R_{\alpha+1,k}\right\} \pm i\sqrt{\frac{(3n-1)}{4m\omega\hbar\alpha(\alpha+1)}}\left\{\sum_{\beta=1}^{\alpha}(P_{\beta k} - \alpha P_{\alpha+1,k}\right\}. \tag{3.44}$$

This relation is preserved if permuting the Cartesian variables:

$$\tilde{a}_{\alpha k}^{\pm} = \sqrt{\frac{(3n-1)m\omega}{4\hbar\alpha(\alpha+1)}}\left\{\sum_{\beta=1}^{\alpha}(\tilde{R}_{\beta k} - \alpha \tilde{R}_{\alpha+1,k}\right\} \pm i\sqrt{\frac{(3n-1)}{4m\omega\hbar\alpha(\alpha+1)}}\left\{\sum_{\beta=1}^{\alpha}(\tilde{P}_{\beta k} - \alpha \tilde{P}_{\alpha+1,k}\right\}. \tag{3.45}$$

Consider the simplest permutation, namely a transposition

$$\tilde{\mathbf{R}}_{A+1} = \mathbf{R}_A,\;\; \tilde{\mathbf{P}}_{A+1} = \mathbf{P}_A,\;\; \tilde{\mathbf{R}}_A = \mathbf{R}_{A+1},\;\; \tilde{\mathbf{P}}_A = \mathbf{P}_{A+1},\;\; \tilde{\mathbf{R}}_C = \mathbf{R}_C,\;\; \tilde{\mathbf{P}}_C = \mathbf{P}_C,\;\; \text{if}\;\; C \neq A, A+1. \tag{3.46}$$

Replacing in (3.45) $\tilde{R}_{Ck}$ and $\tilde{P}_{Ck}$ with R_{Ck} and P_{Ck}, $C = 1, \ldots, n+1$, and expressing the latter through the CAOs from (3.31) and (3.32) we obtain:

$$\tilde{a}_{\alpha k}^{\pm} = \sum_{\beta=1}^{n}(s_{A+1,A})_{\beta\alpha}a_{\beta k}^{\pm}, \tag{3.47}$$

where $s_{A+1,A}$ is $n \times n$ matrix with the following nonzero matrix elements:

$$\begin{aligned}&(s_{A+1,A})_{A-1,A-1} = -(s_{A+1,A})_{A,A} = \frac{1}{A},\;\; (s_{A+1,A})_{\alpha,\alpha} = 1,\;\; \alpha \neq A-1, A;\\ &(s_{A+1,A})_{A-1,A} = (s_{A+1,A})_{A,A-1} = \frac{\sqrt{A^2-1}}{A}.\end{aligned} \tag{3.48}$$

Eq. (3.47) gives the transformation of the CAOs, corresponding to the transposition (3.46).

For any $A < B = 1, \ldots, n+1$ set

$$S_{A,B} = s_{A,A-1}s_{A-1,A-2}s_{A-2,A-3}\ldots s_{2,1}s_{B,B-1}s_{B-1,B-2}s_{B-2,B-3}\ldots s_{3,2}. \tag{3.49}$$

The above matrix leads to a transformation of the CAOs

$$\tilde{a}_{\alpha k}^{\pm} = \sum_{\beta=1}^{n}(S_{A,B})_{\beta\alpha}a_{\beta k}^{\pm}, \tag{3.50}$$

corresponding to a transposition $2 \leftrightarrow B$, followed by $1 \leftrightarrow A$ of the Cartesian variables. Then from (3.34) and (3.50) we derive:

$$(\mathbf{R}_A - \mathbf{R}_B)^2 = \frac{2\hbar}{(3n-1)m\omega} \sum_{i=1}^{3} \{\tilde{a}_{1i}^+, \tilde{a}_{1i}^-\}. \tag{3.51}$$

The matrix $S_{A,B}$ satisfies the requirements of proposition 1: it is symmetric and $S_{A,B}^T S_{A,B} = 1$. Therefore the operators (3.50) satisfy (b) and (c) of proposition 1. Consequently, (b), (3.34) and (3.51) yield:

$$[H, (\mathbf{R}_A - \mathbf{R}_B)^2] = 0, \tag{3.52}$$

whereas from (c) one concludes that the Fock space corresponding to $\tilde{a}_{\alpha k}^{\pm}$ is the same as of $a_{\alpha k}^{\pm}$. Writting (3.35) in the form

$$\frac{2\hbar}{(3n-1)m\omega} \sum_{i=1}^{3} \{a_{1i}^+, a_{1i}^-\} \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (a_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle = \frac{2\hbar}{(3n-1)m\omega} \Big(3p - 3k + \sum_{i=1}^{3} \theta_{1i}\Big) \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (a_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle, \tag{3.53}$$

and applying (e) of proposition 1, we have

$$\frac{2\hbar}{(3n-1)m\omega} \sum_{i=1}^{3} \{\tilde{a}_{1i}^+, \tilde{a}_{1i}^-\} \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle = \frac{2\hbar}{(3n-1)m\omega} \Big(3p - 3k + \sum_{i=1}^{3} \theta_{1i}\Big) \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle,$$

i.e. (see (3.51),

$$(\mathbf{R}_A - \mathbf{R}_B)^2 \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle = \frac{2\hbar}{(3n-1)m\omega} \Big(3p - 3k + \sum_{i=1}^{3} \theta_{1i}\Big) \prod_{\alpha=1}^{n} \prod_{i=1}^{3} (\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle. \tag{3.54}$$

Thus $\prod_{\alpha=1}^{n} \prod_{i=1}^{3} (\tilde{a}_{\alpha i}^+)^{\theta_{\alpha i}} |0\rangle$ are the eigenvectors of $(\mathbf{R}_A - \mathbf{R}_B)^2$ and hence the spectrum of $|\mathbf{R}_A - \mathbf{R}_B|$ is (3.36), i.e., the same as of $|\mathbf{R}_1 - \mathbf{R}_2|$.

The conclusion is that the distances between the particles are quantized. The maximal distance is D as given with (3.37). Hence the space size of the composite system is D. The system exibits a nuclear kind structure: the $n+1-$particles move in a small volume around the centre of mass. Since the coordinates do not commute, the particles are smeared with certain probability within the volume. For any two particles with, say, numbers A and B one can always diagonalize $|\mathbf{R}_A - \mathbf{R}_B|$ simultaneously with the Hamiltonian, i.e., the distance is preserved in time. It is not possible however to diagonalize simultaneously all distance operators, since (see (3.33)) they do not commute.

Let us try to analyze the reason and the amount of the noncommutativity of the coordinates. Since these operators act in the space $W(n,p) = W_{CM} \otimes W(n,p)_{int}$, a more rigorous way to write (3.31) is

$$\mathbf{R}_A = \mathbf{R} \otimes \mathbf{1} + \mathbf{1} \otimes \Big\{ -\sqrt{\frac{A-1}{A}}\, \mathbf{r}_{A-1} + \sum_{\alpha=A}^{n} \sqrt{\frac{1}{\alpha(\alpha+1)}}\, \mathbf{r}_\alpha \Big\}, \tag{3.55}$$

where $\mathbf{1}$ is the unity operator (in the corresponding spase). In the canonical quantum mechanic all operators, having a classical analogue, and in particular the coordinates of the A^{th} particle $\mathbf{R}_A$ are operators only in

W_{CM}, i.e., $\mathbf{R}_A = \mathbf{R} \otimes \mathbf{1}$. The only operator acting nontrivially in $W(n,p)_{int}$ is the spin operator $\mathbf{M}$. In our case, due to the second term in (3.55), also the coordinate operators transform $W(n,p)_{int}$. The second terms are small, they are proportional to $\sqrt{\hbar}$ (see (3.31)), and therefore in a first approximation can be neglacted. If so, then the coordinate operators of all particles $\mathbf{R}_A$ coincide with the centre of mass coordinates $\mathbf{R}$ and the composite system behaves as a canonical point particle with mass $\mu = m(n+1)$. The terms $\left\{ -\sqrt{\frac{A-1}{A}}\, \mathbf{r}_{A-1} + \sum_{\alpha=A}^{n} \sqrt{\frac{1}{\alpha(\alpha+1)}}\, \mathbf{r}_\alpha \right\}$ in (3.55) split the point particle into $n+1$ "pieces", which move in a volume with linear dimension D around the centre of mass. Only within this small volume the coordinates do not commute. To check however this "experimentaly" is nontrivial, since it is imposible to isolate one of the particles, taking it away from the centre of mass.

3.3 A short insight into the algebraic structure

In the present subsection (see also [1]) we discuss shortly the underlying Lie superalgebraical structure of the creation and the annihilation operators (3.8). The presentation is independent from the other part of the paper.

As we have already indicated, any $3n$ pairs of canonical position and momentum operators, namely operators with relations (2.6c), provide the simplest solution of $\mathbf{P1}_{int} - \mathbf{P3}_{int}$. It is not so well known that these operators can be considered as odd generators of a Lie superalgebra. The simplest way to show this is to pass to the related Bose creation and annihilation operators:

$$b_{\alpha k}^{\pm} = \sqrt{\frac{m\omega}{2\hbar}} r_{\alpha k} \mp \frac{i}{\sqrt{2m\omega\hbar}} p_{\alpha k}, \quad \alpha = 1, \ldots, n, \;\; k = 1,2,3. \tag{3.56}$$

It is straightforward to show that the Bose CAOs give one particular representation, the infinite-dimensional Fock representation, of the relations

$$[\{B_{\alpha i}^{\xi}, B_{\beta j}^{\eta}\}, B_{\gamma k}^{\epsilon}] = \delta_{\alpha\gamma}\delta_{ik}(\epsilon-\xi)B_{\beta j}^{\eta} + \delta_{\beta\gamma}\delta_{jk}(\epsilon-\eta)B_{\alpha i}^{\xi},\; i,j,k = 1,2,3,\; \alpha,\beta,\gamma = 1,\ldots,n,\; \xi,\eta,\epsilon = \pm \text{ or } \pm 1. \tag{3.57}$$

Any set of operators $B_{\alpha i}^{\pm}$ with relations (3.57) generate a Lie superalgebra (LS) [14]. It turns out [15] this is the orthosymplectic LS $osp(1/6n)$. The operators $B_{\alpha i}^{\pm}$ are its odd generators, whereas all anticommutators $\{B_{\alpha i}^{\xi}, B_{\beta j}^{\eta}\}$ span the even subalgebra $sp(6n)$. In the terminology of Kac [16] $osp(1/6n)$ is one of the basic Lie superalgebras. Let us mention that in the quantum field theory the operators $B_{\alpha i}^{\pm}$ are known as para-Bose operators. They were introduced by Green as a possible generalization of the statistics of integer spin fields [17].

The creation and the annihilation operators (3.8) generate also a basic Lie superalgebra [1]. Although its relations are similar to (3.57), the algebra is very different. In this case it is the special linear Lie superalgebra $sl(1/3n)$. Its odd generators are the CAOs; all anticommutators $\{a_{\alpha i}^{+}, a_{\beta j}^{-}\}$ span the even subalgebra, which is the Lie algebra $gl(3n)$. This is the reason to call the $(n+1)$-particle system with CAOs (3.8) an $sl(1/3n)$

Wigner quantum system. Resently Okubo has shown that the CAOs (3.8) can be viewed also as generators of a Lie supertriple system [18].

Any representation of the CAOs (3.8) defines a representation of $sl(1/3n)$ and vice versa. Therefore the question to determine the representations of the CAOs (3.8) is equivalent to the problem to construct the representations of $sl(1/3n)$. The hermiticity condition (3.16) defines an antilinear antiinvolution on $sl(1/3n)$. By defininiton the representations in Hilbet spaces, which satisfy (3.16) are called unitary representations (with respect to this antiinvolution). It turns out all such representations are finite-dimensional. They were explicitly constructed in [19] and are labelled with $3n$ numbers, the coordinates of the highest weight. Therefore the Fock representations, considered here, give a small part of all representations, for which the conditions $\mathbf{P1}_{int} - \mathbf{P3}_{int}$ can be satisfied.

Elsewhere we will consider Wigner quantum systems with CAOs generating another basic LS, namely $sl(n/3)$. The LSs $sl(n/3)$ and $sl(1/6n)$ belong to the class **A** in the classification of Kac [16], whereas the algebras $osp(1/6n)$ are from the class **B**. There are two more infinite classes **C** and **D** of basic LSs. It will be interesting to see whether one can introduce CAOs, corresponding to some of them. Certainly one needs not to restrict to solutions, which generate simple LSs. The oscillator conditions $\mathbf{P1}_{int} - \mathbf{P3}_{int}$ can be satisfied with semisimple LSs and in particular with direct sums of LSs as for instance

$$\oplus_{i=1}^{n} sl(1/3) \ \text{ or } \ \oplus_{i=1}^{n} osp(3/2). \tag{3.58}$$

This possibility will be a subject of future considerations.

4. Statistics of the creation and the annihilation operators

Here we discuss shortly the statistics, corresponnding to the algebra of the operators (3.8) and compare it with the very similar statistics of the $g-$ons [10].

To this end we interpret $a_{\alpha i}^{+}$ (resp. $a_{\alpha i}^{-}$) as an operator creating (resp. annihilating) a particle in a (one-particle) state (= orbital) (αi). Then the Pauli principle of the statistics, corresponding to the CAOs (3.8) says that on every orbital there cannot be more that one particle (Fermi-kind property, following from (3.8c)). In addition to this, however, it requires that no more than p orbitals can be simultaneously occupied. The latter is due to the requirement (3.19), namely $\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\theta_{\alpha i} \leq p$. If, for instance, certain p orbitals are occupied, then the possible change $\Delta\theta_{\beta j}$ of the occupation numbers of any other orbital is zero, $\Delta\theta_{\beta j}=0$. Therefore the A-superstatistics is among the exclusion statistics, introduced by Haldane [9]. In fact it is very similar to the statistics of the $g-$ons as defined by Karabali and Nair. We refer to it as to Karabali-Nair statistics (KN-statistics). The latter goes beyond the thermodynamic formulation, attempting a microscopic desctiption of the many-body state space, generated out of a vacuum vector with polynomials of creation and annihilation operators $a_{\alpha i}^{\pm}$ (we keep close to our notation). In order to compare it with the statistics of the CAOs (3.8), we recall the main assumptions of the KN-statistics.

(1) If $|\Theta\rangle \equiv |\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \ldots, \theta_{n1}, \theta_{n2}, \theta_{n3}\rangle$ is a state with $\theta_{\alpha i}$ particles on the orbital (αi), then

$$a_{\alpha i}^{\pm}|\Theta\rangle = c^{\pm}(\Theta)|\Theta\rangle_{\pm\alpha i} \equiv |\theta_{11}, \theta_{12}, \ldots, \theta_{\alpha i} \pm 1, \ldots \theta_{n2}, \theta_{n3}\rangle, \tag{4.1}$$

where $c^{\pm}(\Theta)$ are constants, depending on the statistics.

(2) There exists a number operator $N_{\alpha i}$ of the particles on the orbital (αi), which is a function of $a_{\alpha i}^{+}a_{\alpha i}^{-}$, so that

$$[N_{\alpha i}, a_{\beta j}^{\pm}] = \pm\delta_{\alpha\beta}\delta_{ij}a_{\beta j}^{\pm}. \tag{4.2}$$

Therefore

$$N_{\alpha i}|p, \Theta\rangle = \theta_{ai}|p, \Theta\rangle. \tag{4.3}$$

(3) For any numbers $c_{\alpha i}$ there exists an integer m, so that

$$\Big(\sum_{\alpha=1}^{n}\sum_{i=1}^{3} c_{\alpha i}a_{\alpha i}^{\pm}\Big)^{m+1} = 0. \tag{4.4}$$

(4) The CAOs satisfy the relations $a_{\alpha i}^{-}a_{\beta j}^{-} = R_{\alpha i,\beta j}a_{\beta j}^{-}a_{\alpha i}^{-}$, where $R_{\alpha i,\beta j}$ are numbers.

(5) The CAOs satisfy in addition the relation $[a_{\alpha i}^{+}a_{\alpha i}^{-}, a_{\beta j}^{-}] = 0$.

Clearly the main properties of the Fock representation of the CAOs (3.8) in $W(p,n)$ are very similar to the KN-statistics. Assumption (1) is the same as (3.20). The number operator reads

$$N_{\alpha i} = \frac{p}{3n-1} + \{a_{\alpha i}^{+}, a_{\alpha i}^{-}\} - \frac{1}{3n-1}\sum_{\beta=1}^{n}\sum_{i=1}^{3}\{a_{\beta j}^{+}, a_{\beta j}^{-}\}. \tag{4.5}$$

Therefore (2) also holds, but $N_{\alpha i}$ is not a function only of $a_{\alpha i}^{+}a_{\alpha i}^{-}$, but of all creation and annihilation operators. Assumption (3) holds in our case for $m = p$ and (4) is fulfiled with $R_{\alpha i,\beta j} = -1$. The assumption (5) of the KN-statistics is not satisfied in our case.

Finally, we mention that the creation and the annihilation operators (3.8) (with $n = \infty$) were introduced for the first time in quantum field theory as a possible generalization of the statistics of the tenzor fields [20]. In that case they generate the infinite-dimensional Lie superalgebra $sl(1/\infty)$. The corresponding statistics was called $A-$superstatistics. Recently a representation of the $A-$superstatistics, corresponding to $p = 1$ and called ortho-Fermi statistics was independently proposed by Mishra and Rajasekaran [21].

5. Concluding remarks

The most difficult question to answer in relation to the present approach is why do we interpret the non-canonical operators $\mathbf{R}_\alpha$ and $\mathbf{P}_\alpha$ as coordinates and momenta. A rigorous proof to this question we cannot give. There exists however no proof why the CCRs should necessary hold. This has been noted already by Wigner [5]. All main quantum postulates are satisfied by any WQS. A criterion for accepting or rejecting a

given WQS have to be its predictions and finally the experiment. In this respect some of the predictions of the WQSs are of interest. Quite new, nonconventional feature of the $sl(1/3n)-$quantum system, for instance, is its finite size. The particles move in a small volume around the centre of mass. In a first approximation (see (3.31)), neglecting the terms proportional to $\sqrt{\hbar}$, $\mathbf{R}_\alpha$ and $\mathbf{P}_\alpha$ are canonical, they coincide with $\mathbf{R}$ and $\mathbf{P}$. The noncommutativity of the coordinates and, more generally, the deviation from the CCRs, is due to small, proportional to $\sqrt{\hbar}$, terms added to the CM coordinates and momenta. As a result a point particle of mass $\mu = m(n+1)$ splits into $n+1$ "pieces" with mass m. Only those small, proportional to $\sqrt{\hbar}$, coordinates of the "pieces" with respect to the centre of mass are noncommutative. In this way the canonical point particle is "dressed" with internal structure and it is this "dressing", which is noncanonical. In the limit $\hbar \to 0$ the structure disappears; all $n+1$ "pieces" fall onto the centre of mass. The composite system becomes again a free point particle. It seems to us that such a picture deserves an attention. After all it is unclear so far whether the protons and the neutrons within a nucleus or, say, the constituents of a hadron, the quarks, are canonical.

In answering the above question we could have been also more formal. Nowadays, following the ideas of Connes [22], a lot of work is done in the field of the noncommutative geometry. The quantum groups and the related to them deformed oscillators (see [23] for a list of references) provide other examples of noncanonical quantum systems.

Acknowledgments

We are grateful to Prof. Randjbar-Daemi for the invitation and for the kind hospitality at the High Energy Section of ICTP. The work was supported by the Grant $\Phi - 416$ of the Bulgarian Foundation for Scientific Research.

References

[1] Palev T D 1982 *J. Math. Phys.* **23** 1778; 1982 *Czech. J. Phys. B* **3** 680

[2] Kamupingene A H, Palev T D and Tsaneva S P 1986 *J. Math. Phys.* **27** 2067

[3] Palev T D and Stoilova N I 1994 *J. Phys. A: Math. Gen.* **27** 977

[4] Palev T D and Stoilova N I 1994 *J. Phys. A: Math. Gen.* **27** 7387

[5] Wigner E P 1950 *Phys. Rev.* **77** 711

[6] Ehrenfest P, Z. Phys. 4 (1927) 455

[7] Messia A 1964 *Quantum Mechanics* (North-Holland Publ. Co. Amsterdam)

[8] Ohnuki Y and Kamefuchi S 1978 *J. Math. Phys.* **19** 67; 1979 **Z. Phys. C 2** 367
Okubo S 1980*Phys. Rev.* **D 22** 919
Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L, 1980 *J. Math. Phys.* **21** 2386

Ohnuki and Watanabe S, 1992 *J. Math. Phys.* **33** 3653.

[9] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937

[10] Karabali D and Nair V P 1995 *Nucl. Phys.* **B 438** 551

[11] O'Raifeartaigh L 1970 *Lect. Notes in Phys.* **6** 144 (Editor Bargman V, Springer

[12] Moshinsky M 1969 *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks* (Gordon and Breach Sci. Publ., Berlin-Heidelberg-New York)

[13] Hamermesh M 1960 *Ann. of Phys.* **9** 518

[14] Omote M, Ohnuki Y and Kamefuchi S 1976 *Prog. Theor. Phys.* **56** 1948

[15] Ganchev A Ch and Palev T D 1980 *J. Math. Phys.* **21** 797

[16] Kac V G 1978 *Lect. Notes Math.* **676** 597

[17] Green H S 1953 *Phys. Rev.* **90** 270

[18] Okubo S 1994 *J. Math. Phys.* **35** 2785

[19] Palev T D 1987 *Funct. Anal. Appl.* **21** 245 (English transl.); 1989 *J. Math. Phys.* **30** 1433

[20] Palev T D 1978 *Communication JINR E2-11942*; 1978 *Preprint JINR* **E2-11929, P2-11943**; 1979 *Czech. J. Phys.* **B29** 91; 1980 *J. Math. Phys.* **21** 2560

[21] Mishra A K and Rajasekaran G 1991 *Pramana-J. Phys.* **36** 537
Mishra A K and Rajasekaran G 1992 *Pramana-J. Phys.* **38** L411

[22] Connes A and Lott J 1991 *Nucl. Phys (Proc. Suppl.)* **18B** 29

[23] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)